Multiple zeta functions with arbitrary exponents

1989 J. Phys. A: Math. Gen. 22 931

(http://iopscience.iop.org/0305-4470/22/8/010)

J. Phys. A: Math. Gen. 22 (1989) 931-942. Printed in the UK

Multiple zeta functions with arbitrary exponents

E Elizalde

Department of Structure and Constituents of Matter, Faculty of Physics, University of Barcelona, Diagonal 647, E-08028 Barcelona, Spain

Received 5 May 1988, in final form 26 September 1988

Abstract. Explicit formulae are given for the expression of multiple-sum zeta functions with arbitrary exponents of the type

$$\sum_{n_1,\ldots,n_N=1}^{\infty} (a_1 n_1^{\alpha_1} + \ldots + a_N n_N^{\alpha_N})^{-s}$$

and

$$\sum_{n_1,\ldots,n_N=0}^{\infty} (a_1 n_1^{\alpha_1} + \ldots + a_N n_N^{\alpha_N} + a)^{-s}$$

where a_j, $a > 0$, $j = 1, \ldots, N$, in terms of Riemann and Hurwitz zeta functions.

1. Introduction

Zeta function regularisation of functional determinants is becoming a very important tool in mathematical physics. Let us just mention its uses in quantum field theory in a curved spacetime, in the mathematical calculations involved in string theory, in the computation of anomalies and of effective potentials in the quark confinement problem, and in the evaluation of the partition function for quantum mechanical systems. The number of papers which deal with the zeta function method is increasing rapidly. Here we shall restrict ourselves to the last of the applications just mentioned. Also, the systems to which the methods studied in this paper can be applied will be only the special ones where all the eigenvalues of the Hamiltonian are known exactly.

As described by Actor [1], the partition function for a quantum mechanical system with discrete energy levels $E_n > 0$:

$$Z = \mathrm{Tr}(\mathrm{e}^{-\beta H}) = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta E_n} \tag{1.1}$$

has a high-temperature expansion in terms of the zeta function of H:

$$\zeta_H(s) \equiv \sum_{n=0}^{\infty} E_n^{-s}. \tag{1.2}$$

In the cases generally considered, the series (1.2) is bound to converge absolutely for $\mathrm{Re}(s) > c > 0$ and the zeta function is used in order to analytically continue the series on the RHS of (1.2) to all values of s by a meromorphic function.

In the simplest case when H corresponds to a harmonic oscillator, the energy levels are $E_n = (n+a)\omega$, and one has

$$\zeta_H(s) = \sum_{n=0}^{\infty} (n+a)^{-s}\omega^{-s}. \tag{1.3}$$

For N non-interacting oscillators the Hamiltonian zeta function (1.2) is

$$\zeta_H(s) = \sum_{n_1,\ldots,n_N=0}^{\infty} (n_1\omega_1+\ldots+n_N\omega_N+a)^{-s}. \tag{1.4}$$

These are the more immediate cases. However, during the last few years the study of quantum field theories on partially compactified spacetime manifolds has become important in various domains of quantum physics, such as in the study of dimensional reduction and spontaneous compactification, in the study of quantum field theory in the presence of boundaries (the celebrated Casimir effect) and, in general, in the study of quantum field theory on curved spacetime (manifolds with curvature and non-trivial topology), a step towards quantum gravity.

There are many interesting calculations of effective potentials which can be done exactly and in a very elegant way by using the zeta function method. The usual compactifications studied are the toroidal (spacetime $\mathbb{R}^n \times T^N$) and the spherical (spacetime $\mathbb{R}^n \times S^N$), an important reason for it being that of mathematical simplicity. Noting, however, precludes the possibility of having to consider other compactification manifolds, as string theories seem to indicate. In the case of toroidal compactification one has to deal with expressions of the form (1.4) but quadratic in the n_i. For instance, the zeta function for the vacuum scalar loop on $\mathbb{R}^n \times T^N$ is of the form [1]

$$\zeta_N(s) = A(s) \sum_{n_1,\ldots,n_N=-\infty}^{\infty} [(a_1n_1-b_1)^2+\ldots+(a_Nn_N-b_N)^2+c^2]^{-s+N/2}. \tag{1.5}$$

This zeta function is very difficult to evaluate. In the particular case $a_1=\ldots=a_N$, $c=0$, and $b_1=\ldots=b_N=0$ or $\frac{1}{2}$, it is an Epstein zeta function which, in some cases, can be expressed in terms of Riemann zeta functions. More general cases (in particular, when the a_i are not equal or when c is non-zero) have not been treated because of the great difficulty in evaluating the zeta function.

In the cases of manifolds with non-toroidal compactification the expression of the zeta function is even more involved [1]. So, in the case of spherical compactification, which has also been extensively considered, one has to deal with polynomials in the n_i (with different values for the a_i if the N-sphere turns into an ellipsoid).

As remarked by Actor [1, 2], a basic issue in the zeta function regularisation programme is the evaluation of the above-mentioned types of zeta functions involving multiple summation indices—and depending on several parameters—in terms of the most simple zeta functions available. These are the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{1.6}$$

and the Hurwitz (or generalised Riemann) zeta function:

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}. \tag{1.7}$$

In [1, 2] the binomial expansion has been repeatedly used in order to perform such evaluations. In particular, the following expressions have been obtained:

$$\sum_{m,n=1}^{\infty} (m+bn)^{-s}$$
$$= (b-1)^{-1}(s-1)^{-1}(1-b^{s-1})\zeta(s-1) - \tfrac{1}{2}(1+b^{-s})\zeta(s)$$
$$- \sum_{n=1}^{\infty} \binom{-s}{n} (b-1)^n (b^{-s-n}-1)\zeta(-n)\zeta(s+n) \tag{1.8}$$

valid for $0<b<2$ (with a corresponding expression being valid for $\frac{1}{2}<b<+\infty$), and

$$\sum_{m,n=1}^{\infty} (m+n+c)^{-s} = \zeta(s-1, c) - (1+c)\zeta(s, c) + c^{-s}. \tag{1.9}$$

Actually, this last expression (1.9) had been previously derived in [3] by employing a quite different procedure, which makes use of an integral representation of the generalised Riemann (or Hurwitz) zeta function.

Making repeated use of the binomial expansion and also employing the Euler-Maclaurin formula, the preceding results (1.8) and (1.9) can be easily extended to the evaluation of multiple-sum zeta functions of the form

$$\sum_{n_1,\ldots,n_N=0}^{\infty} (a_1 n_1 + \ldots + a_N n_N + a)^{-s} f(n_i). \tag{1.10}$$

However, the expressions that one obtains in this way are rather lengthy. In § 2 we shall provide alternative formulae for these multiple-sum zeta functions.

Although expression (1.10) is indeed the general form zeta function characteristic of multiple harmonic oscillators, it turns out, however (as has been discussed before), that physical problems formulated in partially compactified spacetimes involve series of the kind

$$\sum_{n_1,\ldots,n_N=1}^{\infty} (a_1 n_1^{\alpha_1} + \ldots + a_N n_N^{\alpha_N})^{-s} f(n_i) \tag{1.11}$$

and

$$\sum_{n_1,\ldots,n_N=0}^{\infty} (a_1 n_1^{\alpha_1} + \ldots + a_N n_N^{\alpha_N} + a)^{-s} f(n_i). \tag{1.12}$$

These series may be viewed as types of generalisations of the Epstein zeta function. As remarked by Actor [1], the only use one can make of the binomial theorem in this case is to express (1.11) and (1.12) in terms of series of the form

$$\sum_{n,m=1}^{\infty} n^{\gamma}(n^{\alpha} + m^{\beta})^{-s}. \tag{1.13}$$

But here neither of the two terms is guaranteed larger than the other and the binomial theorem cannot be used anymore in order to separately evaluate the sums in (1.13) in terms of Riemann zeta functions. This will be resolved in § 3.

2. Zeta functions for non-interacting oscillators

An alternative to the binomial procedure is to make use of our original method [3], which had already led us to (1.9) (in a much simpler case). Let us consider in this section the case of a system of N non-interacting harmonic oscillators. The eigenvalues of the Hamiltonian are

$$a_1 n_1 + \ldots + a_N n_N + a \qquad n_i \in \mathbb{N} \tag{2.1}$$

where the constants $a_i > 0$ are the frequencies ω_i in (1.4). From now on we shall employ this more abstract notation, in line with (1.10)-(1.12). The partition function is

$$\begin{aligned} Z(t) &= \mathrm{Tr}(\mathrm{e}^{-tH}) \\ &= \mathrm{e}^{-at} \sum_{n_1,\ldots,n_N=0}^{\infty} \exp[-t(a_1 n_1 + \ldots + a_N n_N)] \\ &= \mathrm{e}^{-at} \prod_{j=1}^{N} (1-\mathrm{e}^{-a_j t})^{-1}. \end{aligned} \tag{2.2}$$

By making use of the Mellin transform, the zeta function can be written as

$$\begin{aligned} \zeta_H(s) &= \sum_{n_1,\ldots,n_N=0}^{\infty} (a_1 n_1 + \ldots + a_N n_N + a)^{-s} \\ &= \frac{1}{\Gamma(s)} \sum_{n_1,\ldots,n_N=0}^{\infty} \int_0^{\infty} \mathrm{d}t\, t^{s-1} \exp[-t(a_1 n_1 + \ldots + a_N n_N + a)] \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{d}t\, t^{s-1}\, \mathrm{e}^{-at} \prod_{j=1}^{N} (1-\mathrm{e}^{-a_j t})^{-1} \end{aligned} \tag{2.3}$$

for $\mathrm{Re}(s) > N$ and $0 < a < 1$. We shall now develop two different procedures for evaluating the zeta function (2.3).

2.1. *First procedure*

Let us order $a_1 \geq a_2 \geq \ldots \geq a_N$ and expand $Z(t)$ by

$$\begin{aligned} \prod_{j=1}^{N} (1-\mathrm{e}^{-a_j t})^{-1} &= \prod_{j=2}^{N} (1-\mathrm{e}^{-a_j t})^{-1} + \mathrm{e}^{-a_1 t} \prod_{j=1}^{N} (1-\mathrm{e}^{-a_j t})^{-1} = \ldots = (1-\mathrm{e}^{-a_N t})^{-1} \\ &\quad + \sum_{k=1}^{N-1} \mathrm{e}^{-a_k t} \prod_{j=k}^{N} (1-\mathrm{e}^{-a_j t})^{-1}. \end{aligned} \tag{2.4}$$

Dividing once more by a_N, and renaming

$$b_j = a_j / a_N \qquad j = 1, 2, \ldots, N-1 \qquad b = a / a_N \tag{2.5}$$

after substitution of (2.5) into (2.3) and integration, we obtain the following expression:

$$\begin{aligned} \zeta_H(s) = a_N^{-s} \Bigg(& \zeta(s, b) + \sum_{k=1}^{N-1} \zeta(s, b + b_k) + \sum_{1 \leq k \leq h \leq N-1} \zeta(s, b + b_k + b_h) + \ldots \\ & + \sum_{1 \leq k_1 \leq \ldots \leq k_{p-1} \leq N-1} \zeta(s, b + b_{k_1} + \ldots + b_{k_{p-1}}) + R_p(s) \Bigg) \end{aligned} \tag{2.6}$$

with

$$R_p(s)=\frac{1}{\Gamma(s)}\sum_{1\leq k_1\leq\ldots\leq k_p\leq N-1}\int_0^\infty \mathrm{d}t\, t^{s-1}\exp[-(b+b_{k_1}+\ldots+b_{k_p})t]\prod_{j=k_p}^{N}(1-\mathrm{e}^{-b_jt})^{-1}. \tag{2.7}$$

This formal procedure can be rigorously justified as is easily seen from the expression of the remainder (2.7). In fact, let us show that for $\mathrm{Re}\,(s)>2(N-1)$ we have that $R_p\to 0$ as $p\to\infty$. Choose ε such that $0<\varepsilon<1$ and divide $R_p(s)$ into two parts, $R_p^{(1)}(s)$ and $R_p^{(2)}(s)$, corresponding to $\int_0^\infty=\int_0^\varepsilon+\int_\varepsilon^\infty$. We have

$$\begin{aligned}|R_p^{(1)}(s)|&<\frac{K}{\Gamma(s)}p^{N-2}\int_0^\varepsilon \mathrm{d}t\, t^{s-1}\,\mathrm{e}^{-pt}t^{-N}(1+\tfrac{1}{2}Nt)\\ &=\frac{K}{\Gamma(s)}[\Gamma(s-N)p^{-s+2(N-1)}+\tfrac{1}{2}N\Gamma(s-N+1)p^{-s+2N-3}]\to 0\\ &\quad p\to\infty \qquad \mathrm{Re}(s)>2(N-1).\end{aligned} \tag{2.8}$$

Moreover

$$\begin{aligned}|R_p^{(2)}(s)|&<\frac{K}{\Gamma(s)}p^{N-2}(1-\mathrm{e}^{-\varepsilon})^{-N}\int_0^\infty \mathrm{d}t\, t^{s-1}\,\mathrm{e}^{-pt}\\ &=K(1-\mathrm{e}^{-\varepsilon})^{-N}p^{-s+N-2}\to 0 \qquad p\to\infty \qquad \mathrm{Re}(s)>N-2.\end{aligned} \tag{2.9}$$

The series obtained by substitution of (2.4) into (2.3) is absolutely convergent for $\mathrm{Re}(s)>2(N-1)$ and its analytic continuation is thus given by (2.6) for any value of s ($R_p(s)$ being understood as the corresponding analytic continuation of (2.7)).

Summing up, the multidimensional zeta function (2.1) can be written in terms of Hurwitz zeta functions as follows:

$$\begin{aligned}&\sum_{n_1,\ldots,n_N=0}^{\infty}(a_1n_1+\ldots+a_Nn_N+a)^{-s}=a_N^{-s}\sum_{p=0}^{\infty}\sum_{1\leq k_1\leq\ldots\leq k_p\leq N-1}\zeta(s,b+b_{k_1}+\ldots+b_{k_p})\\ &a_1\geq a_2\geq\ldots\geq a_N>0 \qquad a>0 \qquad b_k=a_k/a_N \qquad b=a/a_N.\end{aligned} \tag{2.10}$$

This expression involves an analytic continuation on the variable s. That is, even though the integral representation (2.3) was valid only for $\mathrm{Re}\, s>N$, (2.10) is valid for any value of s.

In the particular case $a_1=\ldots=a_N$, we obtain

$$\sum_{n_1,\ldots,n_N=0}^{\infty}[a_1(n_1+\ldots+n_N)+a]^{-s}=a_1^{-s}\sum_{p=0}^{\infty}\binom{N+p-2}{p}\zeta(s,a/a_1+p) \tag{2.11}$$

which, for $N=2$, can be written as

$$\sum_{m,n=0}^{\infty}[a_1(m+n)+a]^{-s}=a_1^{-s}[\zeta(s-1,a/a_1)-(a/a_1-1)\zeta(s,a/a_1)] \tag{2.12}$$

thus recovering expression (1.9) (see [3]).

2.2. Second procedure

Though very simple and convenient for the treatment of the zeta function (1.4), an important shortcoming of the preceding procedure is its lack of flexibility, namely it cannot be generalised in a natural way in order to treat the cases given by (1.10)-(1.12).

Let us begin by deriving a second expression for the zeta function (1.4) and by extending it then to the case of (1.10). The zeta functions (1.11) and (1.12) will be dealt with by this second method in the next section.

Turning back to (2.3), we shall now interchange the summations over the n_j and k, being the last of the series expansions of the exponentials in (2.3). On doing this, additional terms appear, as has already been proved in the literature (see, for instance, [4]). It is not difficult to provide a direct derivation of the additional terms which appear in our case (2.3). Either by this direct calculation or as a special case of the lemma of the next section, we have that if

$$\begin{aligned} S_1(t) &= \sum_{n=1}^{\infty} \mathrm{e}^{-nt} = \frac{1}{\mathrm{e}^t - 1} \\ \tilde{S}_1(t) &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \sum_{n=1}^{\infty} n^k = \sum_{k=0}^{\infty} \frac{\zeta(-k)}{k!} (-t)^k \end{aligned} \tag{2.13}$$

then

$$S_1(t) = \tilde{S}_1(t) + 1/t \tag{2.14}$$

which coincides with the result obtained in [4] with the help of the Cauchy formula.

Now, by performing the series expansion in (2.3) step by step and by interchanging at each step the order of summation using (2.14), we obtain the following explicit formula (with $b_j = a_j/a_N$, $j = 1, \ldots, N-1$, $b = a/a_N$):

$$\begin{aligned} \zeta_H(s) = \frac{1}{a_N^s \Gamma(s)} \sum_{p=0}^{N-1} \sum_{C_{N-1,p}} \prod_{r=1}^{p} \sum_{k_{j_1}, \ldots, k_{j_r}=0}^{\infty} \left(\frac{(-b_{j_r})^{k_{j_r}}}{k_{j_r}!} \zeta(-k_{j_r}) \right) \\ \times \Gamma\left(s + \sum_{r=1}^{p} k_{j_r} + p + 1 - N \right) \zeta\left(s + \sum_{r=1}^{p} k_{j_r} + p + 1 - N, b \right) \end{aligned} \tag{2.15}$$

where $1 \leqslant j_1 < \ldots < j_p \leqslant N-1$, and where $\Sigma_{C_{N-1,p}}$ means sum over all such selection of p numbers. In all, there are 2^{N-1} terms, and each of them is a multiseries involving the well known Riemann and Hurwitz zeta functions.

The proof that the series one obtains from (2.3) (before analytic continuation) by this second procedure is absolutely convergent is very similar to the one developed before ((2.8) and (2.9)). By estimating the remainder of the series (which, for a product of exponentials, is also of exponential type) and by integrating it in t one shows, as before, that for $\mathrm{Re}(s) > 2N$ the series is absolutely convergent. However, the convergence of the series of analytic continuations is not that immediate in this case. Practical applications of formula (2.15) (in particular to the direct calculation of Casimir energy densities by direct summation over the zero modes) have shown that it is, in fact, convergent in some cases while in others it is only asymptotic. In these latter cases, taking a finite number of terms (≈ 10) we have obtained a very good numerical accuracy ($\approx 10^{-7}$ in relative magnitude) and a remarkable stability of the asymptotic series [5].

This second procedure is immediately generalisable to the expression (1.10) with $f(n_i)$ a polynomial in the n_i (these are actually the cases which come out in practice

[1, 2]). In fact, let us write

$$f(n_i)=\sum_{l_1,\ldots,l_N} c_{l_1\ldots l_N} n_1^{l_1}\ldots n_N^{l_N}. \tag{2.16}$$

Then the result is very similar to (2.15):

$$\tilde{\zeta}_H(s)=\frac{1}{a_N^s\Gamma(s)}\sum_{l_1,\ldots,l_N} c_{l_1\ldots l_N}\sum_{p=0}^{N-1}\sum_{C_{N-1,p}}\prod_{r=1}^{p}\sum_{k_{j_1},\ldots,k_{j_r}=0}^{\infty}\left(\frac{(-b_{j_r})^{k_{j_r}}}{k_{j_r}!}\zeta(-k_{j_r}-l_{j_r})\right)$$
$$\times\Gamma\left(s+\sum_{r=1}^{p}k_{j_r}+p+1-N\right)\zeta\left(s+\sum_{r=1}^{p}k_{j_r}+p+1-N-l_N,b\right). \tag{2.17}$$

Notice, moreover, that the other, more simple, equation (2.9) obtained by the first procedure cannot be generalised to this case where $f(n_i)\neq 1$. Concerning the convergence of the series (2.17) before and after analytic continuation, the same considerations as above apply here.

3. Zeta functions with arbitrary exponents

Let us first recall the toroidal zeta functions, which were studied by Epstein [6, 7]. In the notation of Actor [8], they are

$$Z_N(s)=\sum_{n_1,\ldots,n_N=-\infty}^{\infty}{}' (n_1^2+\ldots+n_N^2)^{-s} \tag{3.1}$$

$$Y_N(s)=\sum_{n_1,\ldots,n_N=-\infty}^{\infty}[(n_1+\tfrac{1}{2})^2+\ldots+(n_N+\tfrac{1}{2})^2]^{-s} \tag{3.2}$$

where the prime means that the term $n_1=\ldots=n_N=0$ is to be omitted in the series (3.1).

Before we attack the general multiple series (1.11) and (1.12), let us concentrate on the case

$$M_2(s;a,b;\alpha,\beta)\equiv\sum_{n,m=1}^{\infty}(an^\alpha+bm^\beta)^{-s}\qquad a,b>0. \tag{3.3}$$

The strategy will be, once more, to make use of the Mellin transform in the zeta function [3]. We get in the present case

$$M_2(s)=\frac{1}{b^s\Gamma(s)}\left[\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{a}{b}\right)^k\zeta(-\alpha k)\int_0^\infty \mathrm{d}t\, t^{s+k-1}\exp(-m^\beta t)\right.$$
$$\left.+\sum_{m=1}^{\infty}\int_0^\infty \mathrm{d}t\, t^{s-1}f_\alpha(t)\exp(-m^\beta t)\right] \tag{3.4}$$

where the second term is the additional term which appears on interchanging the order of summation, i.e.

$$S_\alpha(t)=\tilde{S}_\alpha(t)+f_\alpha(t) \tag{3.5}$$

where

$$S_\alpha(t)=\sum_{n=1}^{\infty}\exp(-n^\alpha t)$$
$$\tilde{S}_\alpha(t)=\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\sum_{n=1}^{\infty}n^{\alpha k}=\sum_{k=0}^{\infty}\frac{\zeta(-\alpha k)}{k!}(-t)^k. \tag{3.6}$$

The following lemma generalises a result due to Weldon [4].

Lemma. The difference $f_\alpha(t)$ between these two series is given by

$$f_\alpha(t)=\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)t^{-1/\alpha}+\Delta_\alpha(t) \tag{3.7}$$

where (i) for $-\infty<\alpha<2$ one has

$$\Delta_\alpha(t)=0 \tag{3.8}$$

(ii) for $\alpha=2$, it is

$$\Delta_2(t)=-\left(\frac{\pi}{t}\right)^{1/2}S_2\left(\frac{\pi^2}{t}\right) \tag{3.9}$$

and (iii) in general, for any value of $\alpha\geqslant 2$, $\Delta_\alpha(t)$ is a small contribution as compared with the main term in $f_\alpha(t)$, (3.7) (for instance, $\Delta_2(1)<10^{-4}$).

Proof. It makes use of complex integration and follows the same steps as in [4]. Consider the function

$$S_\alpha(t,s)\equiv\sum_{n=1}^{\infty}\frac{1}{n^{s+1}}\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}n^{\alpha k}\qquad \alpha\in\mathbb{R}. \tag{3.10}$$

The series $\tilde{S}_\alpha(t)$ above is to be interpreted as the analytic continuation of $S_\alpha(t,s)$ to $s=-1$. Now, notice that $S_\alpha(t,s)$ converges for $\mathrm{Re}(s)>0$ large enough. We can write

$$S_\alpha(t,s)=\sum_{n=1}^{\infty}\frac{1}{n^{s+1}}\oint_C\frac{\mathrm{d}k}{2\pi\mathrm{i}}t^k n^{-\alpha k}\Gamma(k) \tag{3.11}$$

where contour C consists of the straight line $\mathrm{Re}(k)=k_0$, with k_0 fixed, $0<k_0<1$, and of the semicircumference at infinity on the left of this line. Going through the same steps as in the paper by Weldon [4] we end up with

$$S_\alpha(t,s)=\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\zeta(s+1-\alpha k)+\frac{1}{\alpha}\Gamma\left(-\frac{s}{\alpha}\right)t^{-1/\alpha}-\Delta_\alpha(t,s)\qquad s/\alpha\notin\mathbb{N} \tag{3.12}$$

where $\Delta_\alpha(t,s)$ is the contribution of the curved part C_1 of the contour C:

$$\Delta_\alpha(t,s)\equiv\int_{C_1}\frac{\mathrm{d}k}{2\pi\mathrm{i}}\zeta(s+1+\alpha k)\Gamma(k)t^k. \tag{3.13}$$

For $\alpha<2$, it can be seen that this integral vanishes as the radius of the contour tends to infinity. In contrast, for $\alpha=2$, using the standard reflection formula

$$\Gamma\left(\frac{z}{2}\right)\zeta(z)=\pi^{z-1/2}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \tag{3.14}$$

we obtain

$$\Delta_2(t)=\Delta_2(t, s=-1)=\left(\frac{\pi}{t}\right)^{1/2}S\left(\frac{\pi^2}{t}\right). \tag{3.15}$$

This contradicts the affirmation of Weldon that his formula was valid for any $\alpha\in\mathbb{N}$ [4]. In fact, Actor had already noticed that this claim of Weldon was wrong but had not managed to obtain the missing term [9].

Finally, for large α, we find

$$\Delta_\alpha(t)\simeq\left(1-\frac{2}{e}\right)\frac{\pi}{\alpha t}S_\alpha\left[\left(\frac{2\pi}{\alpha t}\right)^\alpha\right]. \tag{3.16}$$

Thus, interpreting $S_\alpha(t)$ (and $\tilde{S}_\alpha(t)$) as the analytic continuation of $S_\alpha(t,s)$ (and of the corresponding expression with summations interchanged) to $s=-1$, we have proved the lemma.

These formulae (3.5) and (3.7) can be viewed as the prescription for zeta function regularisation to be applied to expressions like (1.11) and (1.12). In words the prescription is: (i) use the integral representation in terms of the gamma function, (ii) expand the exponentials in the integrand, (iii) change the order of summation step by step, (iv) zeta-regularise the new series, and (v) add a term $\Gamma(\alpha)/(\alpha t^{1/\alpha})+\Delta_\alpha(t)$ (remember that Δ is a small contribution) in the integrand at each step, which contributes to the following steps. Equation (3.4) gives finally

$$M_2(s;a,b;\alpha,\beta)\simeq\sum_{k=0}^{\infty}\frac{\Gamma(s+k)}{k!\Gamma(s)}\frac{(-a)^k}{b^{s+k}}\zeta(-\alpha k)\zeta(\beta(s+k))$$
$$+\frac{\Gamma(1/\alpha)\Gamma(s-1/\alpha)}{\alpha b^s\Gamma(s)}\zeta(\beta(s-1/\alpha)) \tag{3.17}$$

having neglected the Δ term.

Of course, the roles of a and b and of α and β in this expression can be simultaneously interchanged, i.e.

$$M_2(s;b,a;\beta,\alpha)=M_2(s;a,b;\alpha,\beta). \tag{3.18}$$

Equation (3.17) is the desired expression which gives the zeta function with arbitrary exponents (3.3) in terms of a series of ordinary Riemann zeta functions. It is to be compared with (1.8), which it very much resembles.

Let us now turn to the case

$$M_2^c(s;a,b;\alpha,\beta)\equiv\sum_{n,m=1}^{\infty}(an^\alpha+bm^\beta+c)^{-s}\qquad a,b,c>0. \tag{3.19}$$

By making use again of the integral representation of the zeta function [3] and of the preceding manipulations, we obtain

$$M_2^c(s; a, b; \alpha, \beta) \simeq \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{k!\Gamma(s)} \frac{(-a)^k}{b^{s+k}} \zeta(-\alpha k)[\zeta(\beta(s+k), c/b) - (b/c)^{\beta(s+k)}] + \frac{\Gamma(1/\alpha)\Gamma(s-1/\alpha)}{\alpha b^s \Gamma(s)} [\zeta(\beta(s-1/\alpha), c/b) - (b/c)^{\beta(s-1/\alpha)}]. \qquad (3.20)$$

Let us now develop in detail the three-dimensional case. With two successive applications of the procedure, one gets

$$\begin{aligned}
\sum_{n_1,n_2,n_3=1}^{\infty} & (a_1 n_1^{\alpha_1} + a_2 n_2^{\alpha_2} + a_3 n_3^{\alpha_3})^{-s} \\
&\simeq \frac{1}{a_3^s \Gamma(s)} \Bigg(\sum_{n_2,n_3=1}^{\infty} \sum_{k_1=0}^{\infty} \frac{(-b_1)^{k_1}}{k_1!} \zeta(-\alpha_1 k_1) \\
&\quad \times \int_0^{\infty} \mathrm{d}t\, t^{s+k_1-1} \exp[-t(b_2 n_2^{\alpha_2} + b_3 n_3^{\alpha_3})] \\
&\quad + \frac{\Gamma(1/\alpha_1)}{\alpha_1 b_1^{1/\alpha_1}} \sum_{n_2,n_3=1}^{\infty} \int_0^{\infty} \mathrm{d}t\, t^{s-(1/\alpha_1)-1} \exp[-t(b_2 n_2^{\alpha_2} + b_3 n_3^{\alpha_3})] \Bigg) \\
&= \frac{1}{a_3^s \Gamma(s)} \Bigg(\sum_{n_3=1}^{\infty} \sum_{k_1,k_2=0}^{\infty} \frac{(-b_1)^{k_1}}{k_1!} \frac{(-b_2)^{k_2}}{k_2!} \zeta(-\alpha_1 k_1) \zeta(-\alpha_2 k_2) \\
&\quad \times \int_0^{\infty} \mathrm{d}t\, t^{s+k_1+k_2-1} \exp(-n_3^{\alpha_3} t) \Bigg) \\
&\quad + \frac{\Gamma(1/\alpha_2)}{\alpha_2 b_2^{1/\alpha_2}} \sum_{n_3=1}^{\infty} \sum_{k_1=0}^{\infty} \frac{(-b_1)^{k_1}}{k_1!} \zeta(-\alpha_1 k_1) \int_0^{\infty} \mathrm{d}t\, t^{s+k_1-(1/\alpha_2)-1} \exp(-n_3^{\alpha_3} t) \\
&\quad + \frac{\Gamma(1/\alpha_1)}{\alpha_1 b_1^{1/\alpha_1}} \sum_{n_3=1}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-b_2)^{k_2}}{k_2!} \zeta(-\alpha_2 k_2) \int_0^{\infty} \mathrm{d}t\, t^{s+k_2-(1/\alpha_1)-1} \exp(-n_3^{\alpha_3} t) \\
&\quad + \frac{\Gamma(1/\alpha_1)\Gamma(1/\alpha_2)}{\alpha_1 b_1^{1/\alpha_1} \alpha_2 b_2^{1/\alpha_2}} \sum_{n_3=1}^{\infty} \int_0^{\infty} \mathrm{d}t\, t^{s-(1/\alpha_1)-(1/\alpha_2)-1} \exp(-n_3^{\alpha_3} t) \\
&= \frac{1}{a_3^s \Gamma(s)} \Bigg(\sum_{k_1,k_2=0}^{\infty} \frac{(-b_1)^{k_1}}{k_1!} \frac{(-b_2)^{k_2}}{k_2!} \\
&\quad \times \Gamma(s+k_1+k_2) \zeta(-\alpha_1 k_1) \zeta(-\alpha_2 k_2) \zeta(\alpha_3(s+k_1+k_2)) \Bigg) \\
&\quad + \frac{\Gamma(1/\alpha_2)}{\alpha_2 b_2^{1/\alpha_2}} \sum_{k_1=0}^{\infty} \frac{(-b_1)^{k_1}}{k_1!} \Gamma(s+k_1-1/\alpha_2) \zeta(-\alpha_1 k_1) \zeta(\alpha_3(s+k_1-1/\alpha_2)) \\
&\quad + \frac{\Gamma(1/\alpha_1)}{\alpha_1 b_1^{1/\alpha_1}} \sum_{k_2=0}^{\infty} \frac{(-b_2)^{k_2}}{k_2!} \Gamma(s+k_2-1/\alpha_1) \zeta(-\alpha_2 k_2) \zeta(\alpha_3(s+k_2-1/\alpha_1)) \\
&\quad + \frac{\Gamma(1/\alpha_1)\Gamma(1/\alpha_2)}{\alpha_1 b_1^{1/\alpha_1} \alpha_2 b_2^{1/\alpha_2}} \Gamma(s-1/\alpha_1-1/\alpha_2) \zeta(\alpha_3(s-1/\alpha_1-1/\alpha_2)) \qquad (3.21)
\end{aligned}$$

where $b_j = a_j/a_3$, $j = 1, 2$.

In the general case, for the expressions (1.11) and (1.12) with $f(n_i)=1$, we obtain the explicit formulae

$$M(s; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N)$$

$$\equiv \sum_{n_1,\ldots,n_N=1}^{\infty} (a_1 n_1^{\alpha_1} + \ldots + a_N n_N^{\alpha_N})^{-s}$$

$$\simeq \frac{1}{a_N^s \Gamma(s)} \sum_{p=0}^{N-1} \sum_{C_{N-1,p}} \prod_{r=1}^{p} \frac{1}{\alpha_{i_r}} \Gamma\left(\frac{1}{\alpha_{i_r}}\right)$$

$$\times \sum_{k_{j_1},\ldots,k_{j_{N-p-1}}=0}^{\infty} \Gamma\left(s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_{i_r}}\right)$$

$$\times \prod_{l=1}^{N-p-1} \left[\frac{(-1)^{k_{j_l}}}{k_{j_l}!} \left(\frac{a_{j_l}}{a_N}\right)^{k_{j_l}} \zeta(-\alpha_{j_l} k_{j_l})\right] \zeta\left(\alpha_N \left(s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_{i_r}}\right)\right) \tag{3.22}$$

with $1 \leq i_1 < \ldots < i_p \leq N-1$, $1 \leq j_1 < \ldots < j_{N-p-1} \leq N-1$, being $i_1, \ldots, i_p, j_1, \ldots, j_{N-p-1}$ a permutation of $1, 2, \ldots, N-1$. The sum on $C_{N-1,p}$ means sum over the $\binom{N-1}{p}$ choices of the $i_1, \ldots, i_p$ among $1, \ldots, N-1$.

As in the case considered in § 2 ((2.15) and (2.17)), it is here also immediate to obtain the formula corresponding to (1.11) with $f(n_i)$ of polynomial type (2.16). Moreover, the discussion on the convergence of (2.15) before and after analytic continuation can be transported here in the same terms: the series (3.22) is proven to be absolutely convergent, before analytic continuation, for a value of s with $\mathrm{Re}(s) > 0$ large enough. However, after analytically continuing in s, the series (3.22) is, in general, only asymptotic. As for the applications of these formulae for general α_i we refer the reader back to § 1.

Let us finally consider expressions of the following type, which give a result similar to (3.22):

$$M^c(s; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N)$$

$$\equiv \sum_{n_1,\ldots,n_N=1}^{\infty} (a_1 n_1^{\alpha_1} + \ldots + a_N n_N^{\alpha_N} + c)^{-s}$$

$$\simeq \frac{1}{a_N^s \Gamma(s)} \sum_{p=0}^{N-1} \sum_{C_{N-1,p}} \prod_{r=1}^{p} \frac{1}{\alpha_{i_r}} \Gamma\left(\frac{1}{\alpha_{i_r}}\right)$$

$$\times \sum_{k_{j_1},\ldots,k_{j_{N-p-1}}=0}^{\infty} \Gamma\left(s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{p} \frac{1}{\alpha_{i_r}}\right)$$

$$\times \prod_{l=1}^{N-p-1} \left\{\frac{(-1)^{k_{j_l}}}{k_{j_l}!} \left(\frac{a_{j_l}}{a_N}\right)^{k_{j_l}} \zeta(-\alpha_{j_l} k_{j_l}) \left[\zeta\left(\alpha_N \left(s + \sum_{l=1}^{N-p-1} k_{j_l} - \sum_{r=1}^{\infty} \frac{1}{\alpha_{i_r}}\right), \frac{c}{a_N}\right)\right.\right.$$

$$\left.\left. - \left(\frac{c}{a_N}\right)^{\alpha_N (s + \Sigma_{l=1}^{N-p-1} k_{j_l} - \Sigma_{r=1}^{p} 1/\alpha_{i_r})}\right]\right\}. \tag{3.23}$$

With a little more effort we can also calculate (1.12) (with $f(n_i)=1$, in the polynomial case, the result is very similar):

$$
\begin{aligned}
N^c(s; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N) &\\
\equiv \sum_{n_1,\ldots,n_N=0}^{\infty} & (a_1 n_1^{\alpha_1} + \ldots + a_N n_N^{\alpha_N} + c)^{-s} \\
= M^c(s; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N) &\\
+ \sum_{j=1}^{N} M^c(s; a_1, \ldots, \hat{a}_j, \ldots, a_N; \alpha_1, \ldots, \hat{\alpha}_j, \ldots, \alpha_N) &\\
+ \sum_{\substack{j,k=1 \\ j\neq k}}^{N} M^c(s; a_1, \ldots, \hat{a}_j, \ldots, \hat{a}_k, \ldots, a_N; &\\
\alpha_1, \ldots, \hat{\alpha}_j, \ldots, \hat{\alpha}_k, \ldots, \alpha_N) + \ldots + c^{-s} & \qquad (3.24)
\end{aligned}
$$

where, as customary, the 'hat' over a variable indicates its absence.

Acknowledgments

I am grateful to Professor A Actor for sending his preprints [2, 8] to me prior to publication. I thank the referees of an earlier version of this paper for valuable comments. This work has been partially supported by the Comisión Asesora de Investigación Científica y Técnica (CAICYT, Spain), Research Project AE87-0016-3.

References

[1] Actor A 1987 *J. Phys. A: Math. Gen.* **20** 927, 5351

[2] Actor A 1988 Evaluation of a class of multiple-sum zeta-functions *Preprint* Pennsylvania State University

[3] Elizalde E 1985 *Applications of Field Theory to Statistical Mechanics* (*Lecture Notes in Physics* **216**) ed L Garrido (Berlin: Springer) pp 237–44; 1985 *J. Phys. A: Math. Gen.* **18** 1637; 1986 *Math. Comput.* **47** 347

Elizalde E and Soto J 1985 *Ann. Phys., NY* **162** 192

[4] Weldon H A 1986 *Nucl. Phys.* B **270** 79

[5] Elizalde E 1988 in preparation

[6] Epstein P 1903 *Math. Ann.* **56** 615

[7] Epstein P 1907 *Math. Ann.* **63** 205

[8] Actor A 1988 One-loop Feynman diagram zeta-functions for toroidal spacetime *Preprint* Pennsylvania State University

[9] Actor A 1987 *Fort. Phys.* **35** 793